2.1 QUADRATIC FUNCTIONS AND MODELS

What You Should Learn

- Analyze graphs of quadratic functions.
- Write quadratic functions in standard form and use the results to sketch graphs of functions.
- Find minimum and maximum values of quadratic functions in real-life applications.

The Graph of a Quadratic Function

The Graph of a Quadratic Function

$f(x) = ax + b$ Linear function

$f(x) = c$ Constant function

$f(x) = x^2$ Squaring function

The Graph of a Quadratic Function

Definition of Polynomial Function

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function given by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function of x with degree n.**

The Graph of a Quadratic Function

$f(x) = x^2 + 6x + 2$

$g(x) = 2(x + 1)^2 - 3$

$h(x) = 9 + \frac{1}{4}x^2$

$k(x) = -3x^2 + 4$

$m(x) = (x - 2)(x + 1)$

The Graph of a Quadratic Function

Definition of Quadratic Function

Let a, b, and c be real numbers with $a \neq 0$. The function given by

$$f(x) = ax^2 + bx + c \qquad \text{Quadratic function}$$

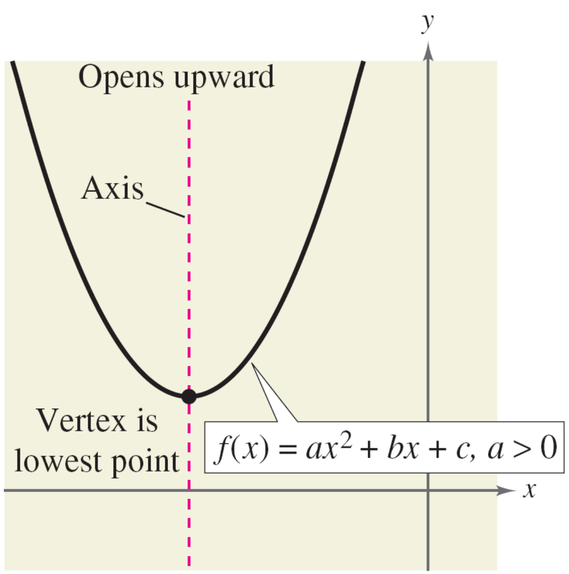
is called a **quadratic function.**

The graph of a quadratic function is a special type of "U"-shaped curve called a **parabola.** Parabolas occur in many real-life applications—especially those involving reflective properties of satellite dishes and flashlight reflectors.

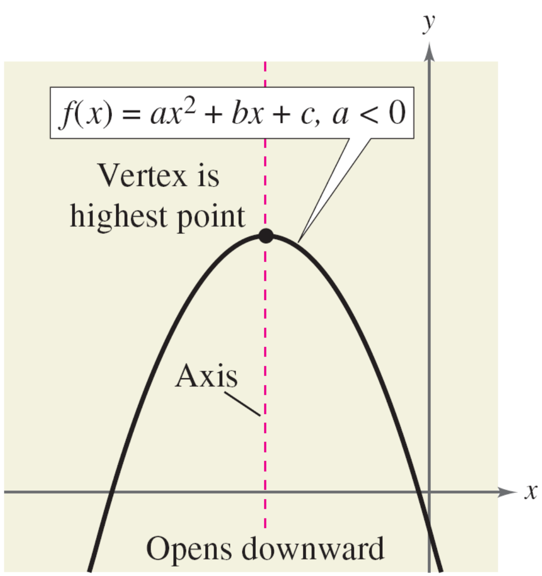
The Graph of a Quadratic Function

All parabolas are symmetric with respect to a line called the **axis of symmetry,** or simply the **axis** of the parabola.

The point where the axis intersects the parabola is the **vertex** of the parabola.



Leading coefficient is positive.



Leading coefficient is negative.

The Graph of a Quadratic Function

$$f(x) = ax^2 + bx + c$$

Leading coefficient $a > 0$

Graph: a parabola opens upward.

$a < 0$

Graph: a parabola opens downward.

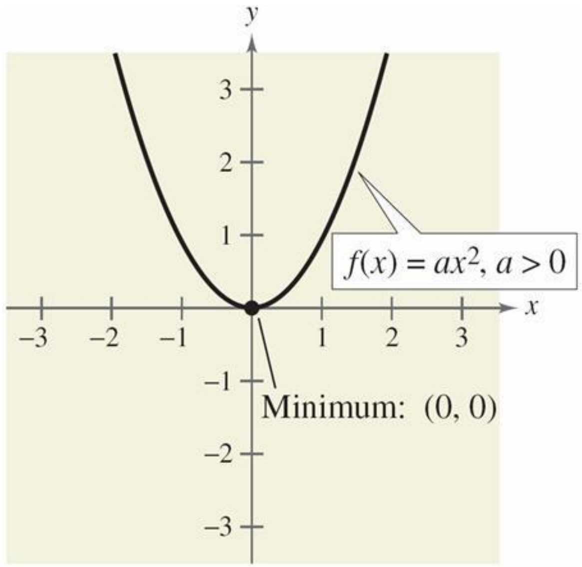
The simplest type of quadratic function is

$$f(x) = ax^2.$$

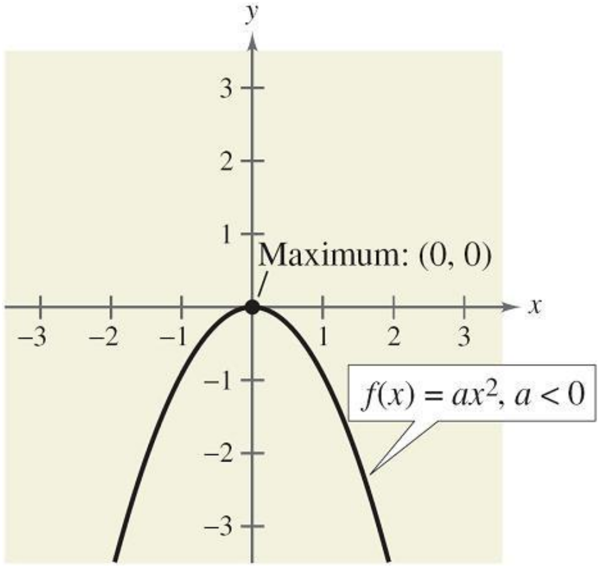
The Graph of a Quadratic Function

If $a > 0$, the vertex is the point with the *minimum* y-value on the graph,

if $a < 0$, the vertex is the point with the *maximum* y-value



Leading coefficient is positive.



Leading coefficient is negative.

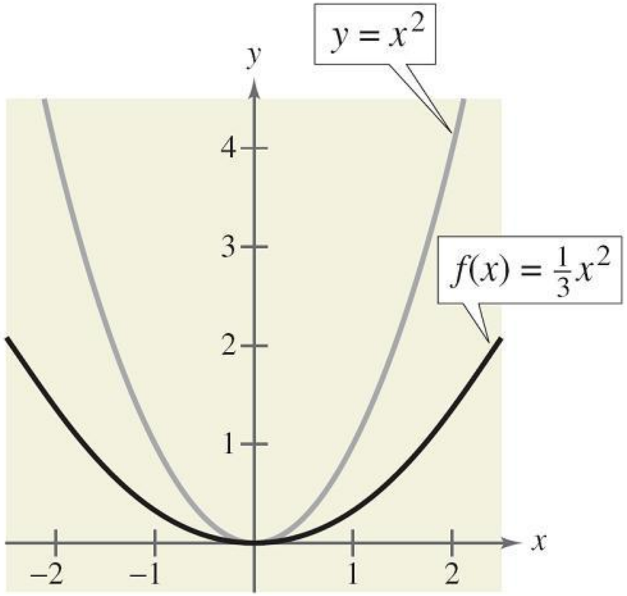
Example 1 – *Sketching Graphs of Quadratic Functions*

a. Compare the graphs of $y = x^2$ and $f(x) = \frac{1}{3}x^2$.

b. Compare the graphs of $y = x^2$ and $g(x) = 2x^2$.

Solution:

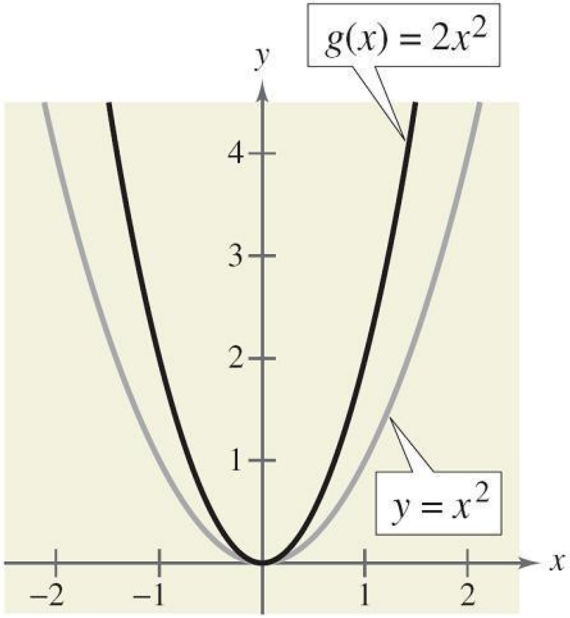
a. Compared with $y = x^2$, each output of $f(x) = \frac{1}{3}x^2$ "shrinks" by a factor of $\frac{1}{3}$, creating the broader parabola shown.



Example 1 – *Solution*

cont'd

b. Compared with $y = x^2$, each output of $g(x) = 2x^2$ "stretches" by a factor of 2, creating the narrower parabola shown.



The Graph of a Quadratic Function

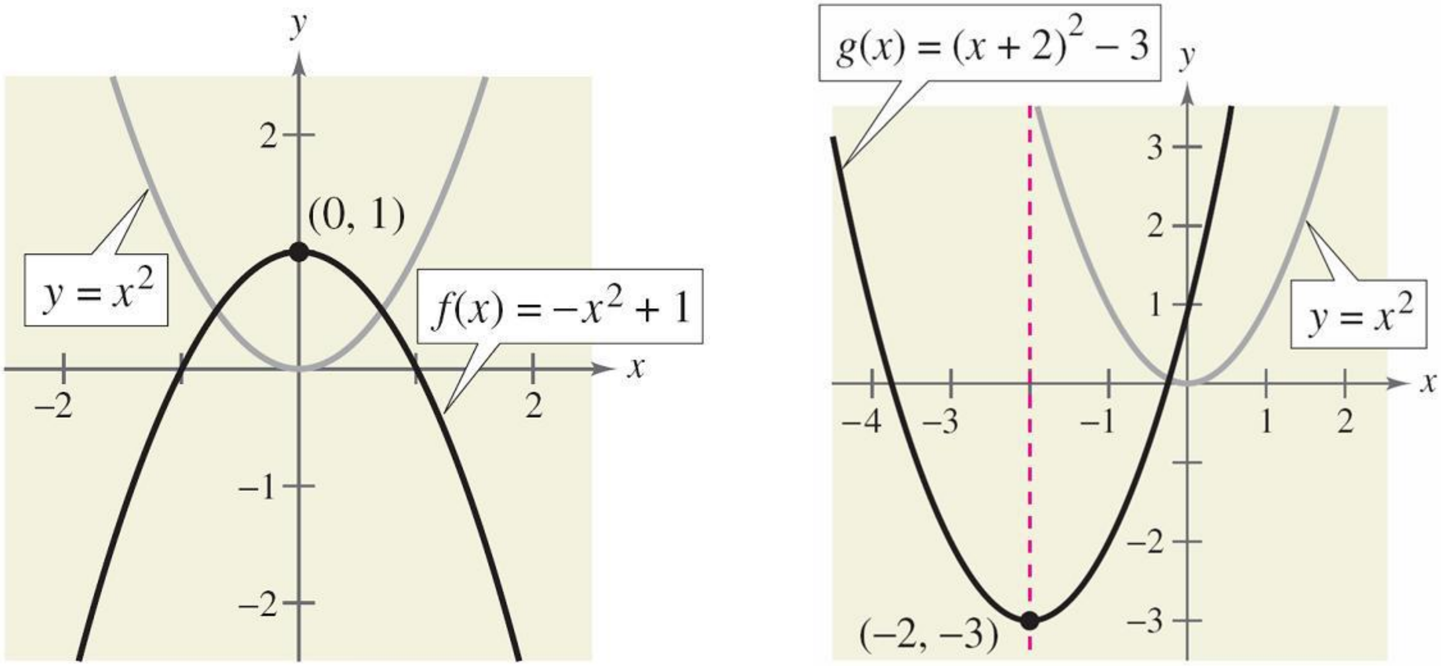
In Example 1, note that the coefficient a determines how widely the parabola given by $f(x) = ax^2$ opens.

If $|a|$ is small, the parabola opens more widely than if $|a|$ is large.

Recall that the graphs of $y = f(x \pm c)$, $y = f(x) \pm c$, $y = f(-x)$, and $y = -f(x)$ are rigid transformations of the graph of $y = f(x)$.

The Graph of a Quadratic Function

For instance, in Figure 2.5, notice how the graph of $y = x^2$ can be transformed to produce the graphs of $f(x) = -x^2 + 1$ and $g(x) = (x + 2)^2 - 3$.



Reflection in *x*-axis followed by an upward shift of one unit

Left shift of two units followed by a downward shift of three units

Figure 2.5

The Standard Form of a Quadratic Function

The Standard Form of a Quadratic Function

Standard Form of a Quadratic Function

The quadratic function given by

$$f(x) = a(x - h)^2 + k, \quad a \neq 0$$

is in **standard form.** The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). If $a > 0$, the parabola opens upward, and if $a < 0$, the parabola opens downward.

Example 2 – *Graphing a Parabola in Standard Form*

Sketch the graph of $f(x) = 2x^2 + 8x + 7$ and identify the vertex and the axis of the parabola.

Solution:

Begin by writing the quadratic function in standard form. Notice that the first step in completing the square is to factor out any coefficient of x^2 that is not 1.

$f(x) = 2x^2 + 8x + 7$ Write original function.

$= 2(x^2 + 4x) + 7$ Factor 2 out of x-terms.

Example 2 – *Solution*

cont'd

$= 2(x^2 + 4x + 4 - 4) + 7$ Add and subtract 4 within parentheses.

$(4/2)^2$

After adding and subtracting 4 within the parentheses, you must now regroup the terms to form a perfect square trinomial.

The –4 can be removed from inside the parentheses; however, because of the 2 outside of the parentheses, you must multiply –4 by 2, as shown below.

$f(x) = 2(x^2 + 4x + 4) - 2(4) + 7$ Regroup terms.

Example 2 – *Solution*

cont'd

$= 2(x^2 + 4x + 4) - 8 + 7$ Simplify.

$= 2(x + 2)^2 - 1$ Write in standard form.

From this form, you can see that the graph of *f* is a parabola that opens upward and has its vertex at (–2, –1).

This corresponds to a left shift of two units and a downward shift of one unit relative to the graph of $y = 2x^2$, as shown in Figure 2.6.

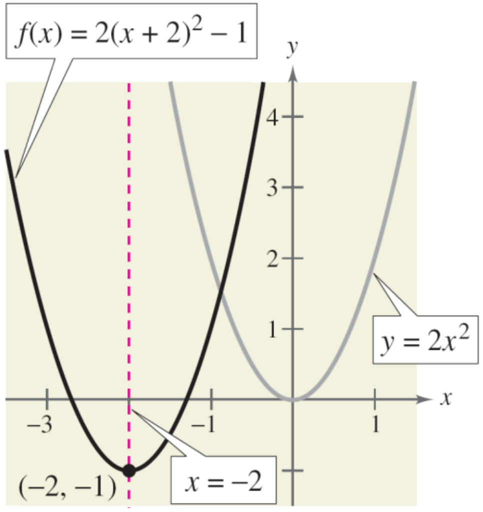


Figure 2.6

Example 2 – *Solution*

cont'd

In the figure, you can see that the axis of the parabola is the vertical line through the vertex, $x = –2$.

The Standard Form of a Quadratic Function

To find the x-intercepts of the graph of $f(x) = ax^2 + bx + c$, you must solve the equation $ax^2 + bx + c = 0$.

If $ax^2 + bx + c$ does not factor, you can use the Quadratic Formula to find the x-intercepts.

Remember, however, that a parabola may not have x-intercepts.

Finding Minimum and Maximum Values

Finding Minimum and Maximum Values

$$f(x) = ax^2 + bx + c$$

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

Standard form

Finding Minimum and Maximum Values

Minimum and Maximum Values of Quadratic Functions

Consider the function $f(x) = ax^2 + bx + c$ with vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

1. If $a > 0$, f has a *minimum* at $x = -\frac{b}{2a}$. The minimum value is $f\left(-\frac{b}{2a}\right)$.

2. If $a < 0$, f has a *maximum* at $x = -\frac{b}{2a}$. The maximum value is $f\left(-\frac{b}{2a}\right)$.

Example 5 – *The Maximum Height of a Baseball*

A baseball is hit at a point 3 feet above the ground at a velocity of 100 feet per second and at an angle of 45° with respect to the ground. The path of the baseball is given by the function $f(x) = –0.0032x^2 + x + 3$, where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from home plate (in feet). What is the maximum height reached by the baseball?

Example 5 – *Solution*

For this quadratic function, you have

$$f(x) = ax^2 + bx + c$$

$$= -0.0032x^2 + x + 3$$

which implies that $a = -0.0032$ and $b = 1$.

Because $a < 0$, the function has a maximum when $x = -b/(2a)$. So, you can conclude that the baseball reaches its maximum height when it is x feet from home plate, where x is

$$x = -\frac{b}{2a}$$

Example 5 – *Solution*

cont'd

$$= -\frac{1}{2(-0.0032)}$$

$$= 156.25 \text{ feet.}$$

At this distance, the maximum height is

$$f(156.25) = -0.0032(156.25)^2 + 156.25 + 3$$

$$= 81.125 \text{ feet.}$$