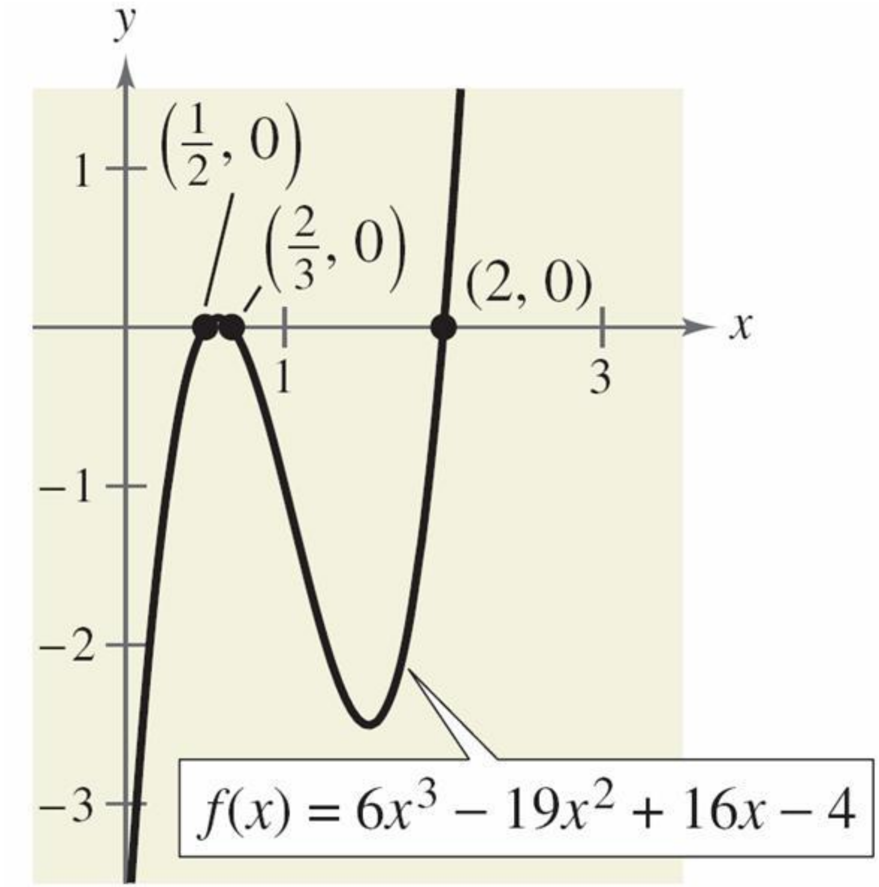
2.3 POLYNOMIAL AND SYNTHETIC DIVISION

What You Should Learn

- Use long division to divide polynomials by other polynomials.
- Use synthetic division to divide polynomials by binomials of the form $(x - k)$.
- Use the Remainder Theorem and the Factor Theorem.

Long Division of Polynomials

Long Division of Polynomials



Example 1 – *Long Division of Polynomials*

Divide $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

Example 1 – *Solution*

Think $\dfrac{6x^3}{x} = 6x^2.$

Think $\dfrac{-7x^2}{x} = -7x.$

Think $\dfrac{2x}{x} = 2.$

$$
\begin{array}{r@{\,}r l}
 & 6x^2 - 7x + 2 & \\
x - 2 & \overline{)\,6x^3 - 19x^2 + 16x - 4} & \\
 & \underline{6x^3 - 12x^2\phantom{{}+ 16x - 4}} & \text{Multiply: } 6x^2(x-2). \\
 & -7x^2 + 16x\phantom{{}- 4} & \text{Subtract.} \\
 & \underline{-7x^2 + 14x\phantom{{}- 4}} & \text{Multiply: } -7x(x-2). \\
 & 2x - 4 & \text{Subtract.} \\
 & \underline{2x - 4} & \text{Multiply: } 2(x-2). \\
 & 0 & \text{Subtract.}
\end{array}
$$

Example 1 – *Solution*

cont'd

From this division, you can conclude that

$6x^3 - 19x^2 + 16x - 4 = (x - 2)(6x^2 - 7x + 2)$

and by factoring the quadratic $6x^2 - 7x + 2$, you have

$6x^3 - 19x^2 + 16x - 4 = (x - 2)(2x - 1)(3x - 2)$.

Example 1 – *Solution*

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Note that this factorization agrees with the graph shown in Figure 2.28 in that the three x-intercepts occur at $x = 2$, $x = \frac{1}{2}$, and $x = \frac{2}{3}$.

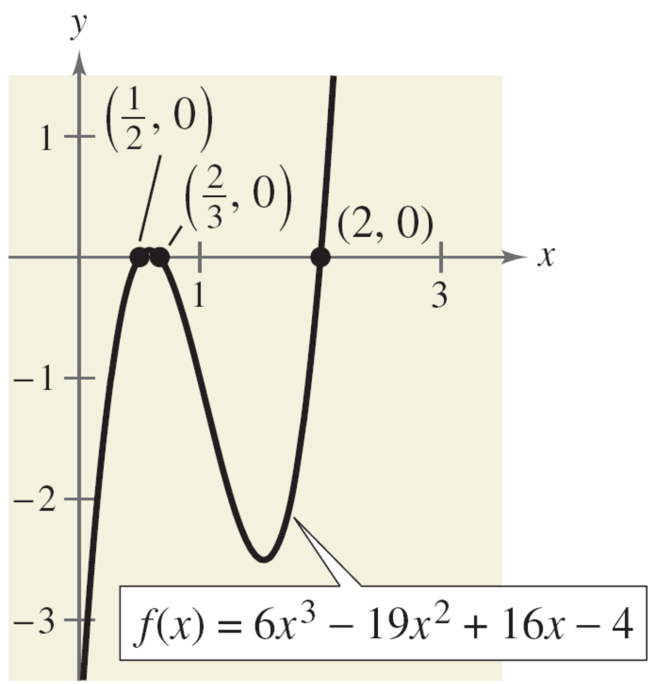


Figure 2.28

Long Division of Polynomials

$$\begin{array}{rrl}
 & x + 2 & \longleftarrow \text{Quotient} \\
\text{Divisor} \longrightarrow \quad x + 1 & \overline{)\, x^2 + 3x + 5} & \longleftarrow \text{Dividend} \\
 & \underline{x^2 + x \quad\;\;\;} & \\
 & 2x + 5 & \\
 & \underline{2x + 2} & \\
 & 3 & \longleftarrow \text{Remainder}
\end{array}$$

Long Division of Polynomials

$$\frac{\overbrace{x^2 + 3x + 5}^{\text{Dividend}}}{\underbrace{x + 1}_{\text{Divisor}}} = \overbrace{x + 2}^{\text{Quotient}} + \frac{\overset{\substack{\text{Remainder} \\ \downarrow}}{3}}{\underbrace{x + 1}_{\text{Divisor}}}$$

This implies that

$x^2 + 3x + 5 = (x + 1)(x + 2) + 3$ Multiply each side by $(x + 1)$.

Long Division of Polynomials

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\underbrace{f(x)}_{\text{Dividend}} = \underbrace{d(x)}_{\text{Divisor}}\underbrace{q(x)}_{\text{Quotient}} + \underbrace{r(x)}_{\text{Remainder}}$$

where $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $d(x)$ *divides evenly* into $f(x)$.

Long Division of Polynomials

The Division Algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

$$\frac{f(x)}{d(x)}$$

improper because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$.

proper because the degree of $r(x)$ is less than the degree of $d(x)$.

Long Division of Polynomials

1. Write the dividend and divisor in descending powers of the variable.

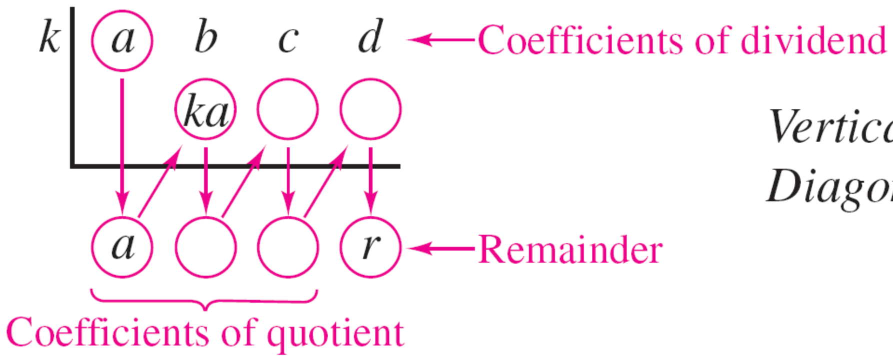
2. Insert placeholders with zero coefficients for missing powers of the variable.

Synthetic Division

Synthetic Division

Synthetic Division (for a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.



Vertical pattern: Add terms.

Diagonal pattern: Multiply by k.

Synthetic Division

This algorithm for synthetic division works only for divisors of the form $x - k$.

Remember that

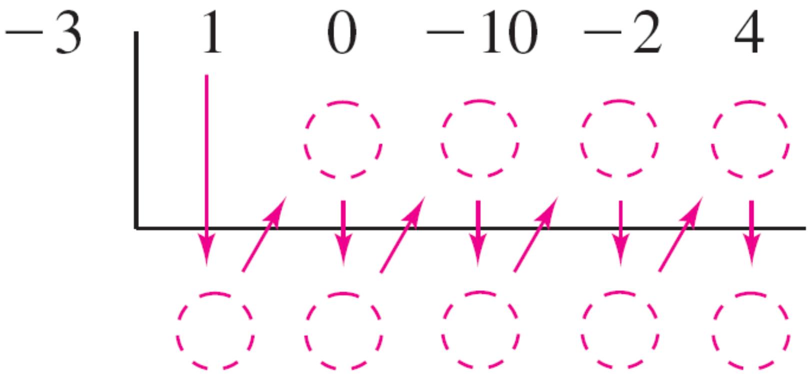
$$x + k = x - (-k).$$

Example 4 – *Using Synthetic Division*

Use synthetic division to divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$.

Solution:

You should set up the array as follows. Note that a zero is included for the missing x^3-term in the dividend.



Example 4 – *Solution*

cont'd

Then, use the synthetic division pattern by adding terms in columns and multiplying the results by –3.

Divisor: $x + 3$ Dividend: $x^4 - 10x^2 - 2x + 4$

$$\begin{array}{r|rrrrr} -3 & 1 & 0 & -10 & -2 & 4 \\ & & -3 & 9 & 3 & -3 \\ \hline & 1 & -3 & -1 & 1 & 1 \end{array}$$

Remainder: 1

Quotient: $x^3 - 3x^2 - x + 1$

So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

The Remainder and Factor Theorems

The Remainder and Factor Theorems

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, the remainder is

$$r = f(k).$$

Example 5 – *Using the Remainder Theorem*

Use the Remainder Theorem to evaluate the following function at $x = -2$.

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

Solution:

Using synthetic division, you obtain the following.

$$\begin{array}{r|rrrr} -2 & 3 & 8 & 5 & -7 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & -9 \end{array}$$

Example 5 – *Solution*

cont'd

Because the remainder is $r = -9$, you can conclude that

$f(-2) = -9.$ $r = f(k)$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

Check:

$$\begin{aligned} f(-2) &= 3(-2)^3 + 8(-2)^2 + 5(-2) - 7 \\ &= 3(-8) + 8(4) - 10 - 7 \\ &= -9 \end{aligned}$$

The Remainder and Factor Theorems

The Factor Theorem

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

Example 6 – *Factoring a Polynomial: Repeated Division*

Show that $(x - 2)$ and $(x + 3)$ are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

Solution:

Using synthetic division with the factor $(x - 2)$, you obtain the following.

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & 0 \end{array} \longrightarrow$$

0 remainder, so $f(2) = 0$ and $(x - 2)$ is a factor.

Example 6 – *Solution*

cont'd

Take the result of this division and perform synthetic division again using the factor (x + 3).

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & \multicolumn{3}{l}{\underbrace{2 \qquad\quad 5 \qquad\quad 3}_{2x^2 + 5x + 3}} & 0 \end{array}$$

0 remainder, so $f(–3) = 0$ and (x + 3) is a factor.

Because the resulting quadratic expression factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x – 2)(x + 3)(2x + 3)(x + 1).$$

The Remainder and Factor Theorems

Uses of the Remainder in Synthetic Division

The remainder r, obtained in the synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.

2. If $r = 0$, $(x - k)$ is a factor of $f(x)$.

3. If $r = 0$, $(k, 0)$ is an x-intercept of the graph of f.