What You Should Learn

• Use the Fundamental Theorem of Algebra to determine the number of zeros of polynomial functions.

• Find rational zeros of polynomial functions.

• Find conjugate pairs of complex zeros.

• Find zeros of polynomials by factoring.

• Use Descartes’s Rule of Signs and the Upper and Lower Bound Rules to find zeros of polynomials.
The Fundamental Theorem of Algebra
The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree $n$, where $n > 0$, then $f$ has at least one zero in the complex number system.

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree $n$, where $n > 0$, then $f$ has precisely $n$ linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.
Example 1 – Zeros of Polynomial Functions

a. The first-degree polynomial $f(x) = x - 2$ has exactly one zero: $x = 2$.

b. Counting multiplicity, the second-degree polynomial function

$$f(x) = x^2 - 6x + 9$$

$$= (x - 3)(x - 3)$$

has exactly two zeros: $x = 3$ and $x = 3$ (repeated zero)
c. The third-degree polynomial function

\[ f(x) = x^3 + 4x \]

\[ = x(x^2 + 4) \]

\[ = x(x - 2i)(x + 2i) \]

has exactly three zeros: \( x = 0 \), \( x = 2i \), and \( x = -2i \).
Example 1 – Zeros of Polynomial Functions

\textbf{d.} The fourth-degree polynomial function

\[ f(x) = x^4 - 1 \]

\[ = (x - 1)(x + 1)(x - i)(x + i) \]

has exactly \textit{four} zeros: \( x = 1, \ x = -1, \ x = i, \ \text{and} \ x = -i. \)
The Rational Zero Test
The Rational Zero Test

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ has integer coefficients, every rational zero of $f$ has the form

$$\text{Rational zero} = \frac{p}{q}$$

where $p$ and $q$ have no common factors other than 1, and

$p = \text{a factor of the constant term } a_0$

$q = \text{a factor of the leading coefficient } a_n$.

Possible rational zeros $= \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$
Example 3 – Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of \( f(x) = x^4 - x^3 + x^2 - 3x - 6 \).

**Solution:**
Because the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

*Possible rational zeros: \( \pm 1, \pm 2, \pm 3, \pm 6 \)*
Example 3 – Solution

So, \( f(x) \) factors as

\[
f(x) = (x + 1)(x - 2)(x^2 + 3).
\]
Example 3 – Solution

Figure 2.32

\[ f(x) = x^4 - x^3 + x^2 - 3x - 6 \]
The Rational Zero Test

If the leading coefficient of a polynomial is not 1,

1. * a programmable calculator
2. a graph: good estimate of the locations of the zeros;
3. the Intermediate Value Theorem along with a table
4. synthetic division can be used to test the possible rational zeros.
Example 5

Find all the real zeros of \( f(x) = -10x^3 + 15x^2 + 16x - 12 \)
Conjugate Pairs
Conjugate Pairs

The pairs of complex zeros of the form $a + bi$ and $a - bi$ are conjugates.

Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has real coefficients. If $a + bi$, where $b \neq 0$, is a zero of the function, the conjugate $a - bi$ is also a zero of the function.
Example 7 Finding the Zeros of a Polynomial Functions

Find all the zeros of \( f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60 \) given that \( 1 + 3i \) is a zero of \( f \).
Example 6 – Finding a Polynomial with Given Zeros

Find a fourth-degree polynomial function with real coefficients that has \(-1, -1, \) and \(3i\) as zeros.

Solution:

Because \(3i\) is a zero and the polynomial is stated to have real coefficients, conjugate \(-3i\) must also be a zero.

So, from the Linear Factorization Theorem, \(f(x)\) can be written as

\[
f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).
\]
Example 6 – Solution

For simplicity, let $a = 1$ to obtain

$$f(x) = (x^2 + 2x + 1)(x^2 + 9)$$

$$= x^4 + 2x^3 + 10x^2 + 18x + 9.$$
Factoring a Polynomial
Factoring a Polynomial

\[ f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdots (x - c_n) \]

Factors of a Polynomial

Every polynomial of degree \( n > 0 \) with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.
Factoring a Polynomial

A quadratic factor with no real zeros is said to be *prime* or *irreducible* over the reals.

\[ x^2 + 1 = (x - i)(x + i) \] is irreducible over the reals (and therefore over the rationals).

\[ x^2 - 2 = \left( x - \sqrt{2} \right) \left( x + \sqrt{2} \right) \] is irreducible over the rationals but *reducible* over the reals.
Example 7 – Finding the Zeros of a Polynomial Function

Find all the zeros of \( f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60 \) given that \( 1 + 3i \) is a zero of \( f \).

Solution:

Because complex zeros occur in conjugate pairs, you know that \( 1 - 3i \) is also a zero of \( f \).

Both \([x - (1 + 3i)]\) and \([x - (1 - 3i)]\) are factors of \( f \).

\[
[x - (1 + 3i)][x - (1 - 3i)]
= [(x - 1) - 3i][(x - 1) + 3i]
= (x - 1)^2 - 9i^2
= x^2 - 2x + 10.
\]
Example 7 – Solution

\[ x^2 - 2x + 10 \left( x^4 - 3x^3 + 6x^2 + 2x - 60 \right) \]

\[ x^4 - 2x^3 + 10x^2 \]

\[ -x^3 - 4x^2 + 2x \]

\[ -x^3 + 2x^2 - 10x \]

\[ -6x^2 + 12x - 60 \]

\[ -6x^2 + 12x - 60 \]

0
Example 7 – Solution

\[ f(x) = (x^2 - 2x + 10)(x^2 - x - 6) \]

\[ = (x^2 - 2x + 10)(x - 3)(x + 2) \]

The zeros of \( f \) are \( x = 1 + 3i, x = 1 - 3i, x = 3, \) and \( x = -2. \)
Other Tests for Zeros of Polynomials
Other Tests for Zeros of Polynomials

**Descartes’s Rule of Signs**

Let \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \) be a polynomial with real coefficients and \( a_0 \neq 0 \).

1. The number of *positive real zeros* of \( f \) is either equal to the number of variations in sign of \( f(x) \) or less than that number by an even integer.

2. The number of *negative real zeros* of \( f \) is either equal to the number of variations in sign of \( f(-x) \) or less than that number by an even integer.

A *variation in sign* means that two consecutive coefficients have opposite signs.
When using Descartes’s Rule of Signs, a zero of multiplicity \( k \) should be counted as \( k \) zeros.

\[
x^3 - 3x + 2 = (x - 1)(x - 1)(x + 2)
\]

the two positive real zeros are \( x = 1 \) of multiplicity 2.
Example 9 – *Using Descartes’s Rule of Signs*

Describe the possible real zeros of

\[ f(x) = 3x^3 - 5x^2 + 6x - 4. \]

**Solution:**

The original polynomial has *three* variations in sign.

\[ + \rightarrow - \quad + \rightarrow - \quad + \rightarrow - \]

\[ f(x) = 3x^3 - 5x^2 + 6x - 4. \]

\[ - \rightarrow + \quad - \rightarrow + \]
Example 9 – Solution

The polynomial

\[ f(-x) = 3(-x)^3 - 5(-x)^2 + 6(-x) - 4 \]

\[ = -3x^3 - 5x^2 - 6x - 4 \]

has no variations in sign.

So, from Descartes’s Rule of Signs, the polynomial

\[ f(x) = 3x^3 - 5x^2 + 6x - 4 \]

has either three positive real zeros or one positive real zero, and has no negative real zeros.
Example 9 – Solution

\[ f(x) = 3x^3 - 5x^2 + 6x - 4 \]
Upper and Lower Bound Rules

Let \( f(x) \) be a polynomial with real coefficients and a positive leading coefficient. Suppose \( f(x) \) is divided by \( x - c \), using synthetic division.

1. If \( c > 0 \) and each number in the last row is either positive or zero, \( c \) is an \textbf{upper bound} for the real zeros of \( f \).

2. If \( c < 0 \) and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), \( c \) is a \textbf{lower bound} for the real zeros of \( f \).
1. If the terms of \( f(x) \) have a common monomial factor, it should be factored out before applying the tests in this section. For instance, by writing

\[
f(x) = x^4 - 5x^3 + 3x^2 + x
\]

\[
= x(x^3 - 5x^2 + 3x + 1)
\]

you can see that \( x = 0 \) is a zero of \( f \) and that the remaining zeros can be obtained by analyzing the cubic factor.
2. If you are able to find all but two zeros of \( f(x) \), you can always use the Quadratic Formula on the remaining quadratic factor. For instance, if you succeeded in writing

\[
f(x) = x^4 - 5x^3 + 3x^2 + x
\]

\[
= x(x - 1)(x^2 - 4x - 1)
\]

you can apply the Quadratic Formula to \( x^2 - 4x - 1 \) to conclude that the two remaining zeros are \( x = 2 + \sqrt{5} \) and \( x = 2 - \sqrt{5} \).