4.4 TRIGONOMETRIC FUNCTIONS OF ANY ANGLE
What You Should Learn

• Evaluate trigonometric functions of any angle.

• Find reference angles.

• Evaluate trigonometric functions of real numbers.
Introduction
Definitions of Trigonometric Functions of Any Angle

Let $\theta$ be an angle in standard position with $(x, y)$ a point on the terminal side of $\theta$ and $r = \sqrt{x^2 + y^2} \neq 0$.

\[
\begin{align*}
\sin \theta &= \frac{y}{r} \\
\cos \theta &= \frac{x}{r} \\
\tan \theta &= \frac{y}{x}, \quad x \neq 0 \\
\cot \theta &= \frac{x}{y}, \quad y \neq 0 \\
\sec \theta &= \frac{r}{x}, \quad x \neq 0 \\
\csc \theta &= \frac{r}{y}, \quad y \neq 0
\end{align*}
\]
Example 1 – Evaluating Trigonometric Functions

Let \((-3, 4)\) be a point on the terminal side of \(\theta\). Find the sine, cosine, and tangent of \(\theta\).

Solution:
\(x = -3, \ y = 4,\)

\[
r = \sqrt{x^2 + y^2}
\]

\[
= \sqrt{(-3)^2 + 4^2}
\]

\[
= \sqrt{25}
\]

\[
= 5.
\]
Example 1 – Solution

\[
\sin \theta = \frac{y}{r} = \frac{4}{5},
\]
\[
\cos \theta = \frac{x}{r} = -\frac{3}{5},
\]
\[
\tan \theta = \frac{y}{x} = -\frac{4}{3}.
\]
Introduction
Reference Angles
Definition of Reference Angle

Let $\theta$ be an angle in standard position. Its reference angle is the acute angle $\theta'$ formed by the terminal side of $\theta$ and the horizontal axis.
Reference Angles

The reference angles for $\theta$ in Quadrants II, III, and IV.

- Quadrant II: $\theta' = \pi - \theta$ (radians)
  $\theta' = 180^\circ - \theta$ (degrees)

- Quadrant III: $\theta' = \theta - \pi$ (radians)
  $\theta' = \theta - 180^\circ$ (degrees)

- Quadrant IV: $\theta' = 2\pi - \theta$ (radians)
  $\theta' = 360^\circ - \theta$ (degrees)
Example 4 – Finding Reference Angles

Find the reference angle $\theta'$. 

a. $\theta = 300^\circ$

b. $\theta = 2.3$

c. $\theta = -135^\circ$
Example 4(a) – Solution

Because 300° lies in Quadrant IV, the angle it makes with the x-axis is

\[ \theta' = 360° - 300° \]

= 60°. Degrees
Example 4(b) – Solution

Because 2.3 lies between $\pi/2 \approx 1.5708$ and $\pi \approx 3.1416$, it follows that it is in Quadrant II and its reference angle is

$$\theta' = \pi - 2.3$$

$$\approx 0.8416.$$  

Radians
Example 4(c) – Solution

First, determine that $-135^\circ$ is coterminal with $225^\circ$, which lies in Quadrant III. So, the reference angle is

$$\theta' = 225^\circ - 180^\circ$$

$$= 45^\circ.$$

Degrees

225° and $-135^\circ$ are coterminal.

$\theta' = 45^\circ$

$\theta = -135^\circ$
Trigonometric Functions of Real Numbers
By definition, you know that

\[ \sin \theta = \frac{y}{r} \quad \text{and} \quad \tan \theta = \frac{y}{x}. \]

For the right triangle with acute angle \( \theta' \) and sides of lengths \( |x| \) and \( |y| \), you have

\[ \sin \theta' = \frac{\text{opp}}{\text{hyp}} = \frac{|y|}{r} \]

and

\[ \tan \theta' = \frac{\text{opp}}{\text{adj}} = \frac{|y|}{|x|}. \]
Evaluating Trigonometric Functions of Any Angle

To find the value of a trigonometric function of any angle $\theta$:

1. Determine the function value for the associated reference angle $\theta'$.
2. Depending on the quadrant in which $\theta$ lies, affix the appropriate sign to the function value.

<table>
<thead>
<tr>
<th>$\theta$ (degrees)</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
<th>180°</th>
<th>270°</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$ (radians)</td>
<td>0</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{2}$</td>
</tr>
<tr>
<td>sin $\theta$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>cos $\theta$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
<td>tan $\theta$</td>
<td>0</td>
<td>$\frac{\sqrt{3}}{3}$</td>
<td>1</td>
<td>$\sqrt{3}$</td>
<td>Undef.</td>
<td>0</td>
<td>Undef.</td>
</tr>
</tbody>
</table>
Example 5 – Using Reference Angles

Evaluate each trigonometric function.

a. \( \cos \frac{4\pi}{3} \)

b. \( \tan(-210^\circ) \)

c. \( \csc \frac{11\pi}{4} \)
Because $\theta = 4\pi/3$ lies in Quadrant III, the reference angle is

$$\theta' = \frac{4\pi}{3} - \pi$$

$$= \frac{\pi}{3}$$

as shown in Figure 4.44.

Moreover, the cosine is negative in Quadrant III, so

$$\cos \frac{4\pi}{3} = (-) \cos \frac{\pi}{3}$$

$$= -\frac{1}{2}.$$
Example 5(b) – Solution

Because \(-210^\circ + 360^\circ = 150^\circ\), it follows that \(-210^\circ\) is coterminal with the second-quadrant angle \(150^\circ\).

So, the reference angle is \(\theta' = 180^\circ - 150^\circ = 30^\circ\), as shown in Figure 4.45.
Example 5(b) – Solution

Finally, because the tangent is negative in Quadrant II, you have

\[ \tan(-210^\circ) = (-) \tan 30^\circ \]

\[ = -\frac{\sqrt{3}}{3}. \]
Example 5(c) – Solution

Because \((11\pi/4) - 2\pi = 3\pi/4\), it follows that \(11\pi/4\) is coterminal with the second-quadrant angle \(3\pi/4\).

So, the reference angle is \(\theta' = \pi - (3\pi/4) = \pi/4\), as shown in Figure 4.46.

![Figure 4.46](image-url)
Example 5(c) – Solution

Because the cosecant is positive in Quadrant II, you have

\[
csc \frac{11\pi}{4} = (+) \csc \frac{\pi}{4}
\]

\[
= \frac{1}{\sin(\pi/4)}
\]

\[
= \sqrt{2}.
\]