7.2 TWO-VARIABLE LINEAR SYSTEMS
What You Should Learn

• Use the method of elimination to solve systems of linear equations in two variables.

• Interpret graphically the numbers of solutions of systems of linear equations in two variables.

• Use systems of linear equations in two variables to model and solve real-life problems.
The Method of Elimination
The Method of Elimination

**Method of Elimination**

To use the **method of elimination** to solve a system of two linear equations in \( x \) and \( y \), perform the following steps.

1. *Obtain coefficients* for \( x \) (or \( y \)) that differ only in sign by multiplying all terms of one or both equations by suitably chosen constants.

2. *Add* the equations to eliminate one variable.

3. *Solve* the equation obtained in Step 2.

4. *Back-substitute* the value obtained in Step 3 into either of the original equations and solve for the other variable.

5. *Check* that the solution satisfies *each* of the original equations.
Example 2 – Solving a System of Equations by Elimination

Solve the system of linear equations.

\[
\begin{align*}
2x - 4y &= -7 \\
5x + y &= -1
\end{align*}
\]

Solution:

Multiplying Equation 2 by 4.

\[
\begin{align*}
2x - 4y &= -7 \\
20x + 4y &= -4
\end{align*}
\]

Add equations.

\[
22x = -11
\]

Solve for \(x\).

\[
x = -\frac{1}{2}
\]
Example 2 – Solution

By back-substituting $x = -\frac{1}{2}$ into Equation 1, you can solve for $y$.

$$2x - 4y = -7$$

$$2\left(-\frac{1}{2}\right) - 4y = -7$$

$$-4y = -7$$

$$y = \frac{3}{2}$$

The solution is $\left(-\frac{1}{2}, \frac{3}{2}\right)$. 

Write Equation 1.

Substitute $-\frac{1}{2}$ for $x$.

Combine like terms.

Solve for $y$. 

Example 2 – Solution

Check this in the original system, as follows.

**Check**

\[ 2x - 4y = -7 \]

\[ 2\left(-\frac{1}{2}\right) - 4\left(\frac{3}{2}\right) = \frac{?}{-7} \]

\[-1 - 6 = -7 \]

\[ 5x + y = -1 \]

\[ 5\left(-\frac{1}{2}\right) + \frac{3}{2} = \frac{?}{-1} \]

\[-\frac{5}{2} + \frac{3}{2} = -1 \]

Write original Equation 1.

Substitute into Equation 1.

Equation 1 checks. ✓

Write original Equation 2.

Substitute into Equation 2.

Equation 2 checks. ✓
The Method of Elimination

Equivalent systems: have precisely the same solution set.

\[
\begin{align*}
2x - 4y &= -7 \\
5x + y &= -1
\end{align*}
\quad \text{and} \quad
\begin{align*}
2x - 4y &= -7 \\
20x + 4y &= -4
\end{align*}
\]
The Method of Elimination

The operations that can be performed on a system of linear equations to produce an equivalent system are

(1) interchanging any two equations,
(2) multiplying an equation by a nonzero constant, and
(3) adding a multiple of one equation to any other equation in the system.
Graphical Interpretation of Solutions
### Graphical Interpretations of Solutions

For a system of two linear equations in two variables, the number of solutions is one of the following:

<table>
<thead>
<tr>
<th>Number of Solutions</th>
<th>Graphical Interpretation</th>
<th>Slopes of Lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Exactly one solution</td>
<td>The two lines intersect at one point.</td>
<td>The slopes of the two lines are not equal.</td>
</tr>
<tr>
<td>2. Infinitely many solutions</td>
<td>The two lines coincide (are identical).</td>
<td>The slopes of the two lines are equal.</td>
</tr>
<tr>
<td>3. No solution</td>
<td>The two lines are parallel.</td>
<td>The slopes of the two lines are equal.</td>
</tr>
</tbody>
</table>
A system of linear equations is **consistent** if it has at least one solution.

A consistent system with exactly one solution is *independent*, whereas a consistent system with infinitely many solutions is *dependent*.

A system is **inconsistent** if it has no solution.
Example

\[ \begin{cases} 
5x + 3y = 9 \\
2x - 4y = 14 
\end{cases} \]
Example

\[
\begin{cases}
  x - 2y = 3 \\
  -2x + 4y = 1
\end{cases}
\]
Example

\[
\begin{align*}
2x - y &= 1 \\
4x - 2y &= 3
\end{align*}
\]
In Class Assignment

\[
\begin{align*}
0.2x - 0.5y &= -27.8 \\
0.3x + 0.4y &= 68.7
\end{align*}
\]
Example 4 – Recognizing Graphs of Linear Systems

Match each system of linear equations with its graph

**a.** \[
\begin{align*}
2x - 3y &= 3 \\
-4x + 6y &= 6
\end{align*}
\]

**b.** \[
\begin{align*}
2x - 3y &= 3 \\
x + 2y &= 5
\end{align*}
\]

**c.** \[
\begin{align*}
2x - 3y &= 3 \\
-4x + 6y &= -6
\end{align*}
\]
Example 4 – Solution

a. The graph of system (a) is a pair of parallel lines (ii). The lines have no point of intersection, so the system has no solution. The system is inconsistent.

b. The graph of system (b) is a pair of intersecting lines (iii). The lines have one point of intersection, so the system has exactly one solution. The system is consistent.

c. The graph of system (c) is a pair of lines that coincide (i). The lines have infinitely many points of intersection, so the system has infinitely many solutions. The system is consistent.
Applications
At this point, you may be asking the question “How can I tell which application problems can be solved using a system of linear equations?” The answer comes from the following considerations.

1. Does the problem involve more than one unknown quantity?

2. Are there two (or more) equations or conditions to be satisfied?

If one or both of these situations occur, the appropriate mathematical model for the problem may be a system of linear equations.
Example 8 – An Application of a Linear System

An airplane flying into a headwind travels the 2000-mile flying distance between Chicopee, Massachusetts and Salt Lake City, Utah in 4 hours and 24 minutes. On the return flight, the same distance is traveled in 4 hours. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

Solution:
The two unknown quantities are the speeds of the wind and the plane.
If $r_1$ is the speed of the plane and $r_2$ is the speed of the wind, then

\[ r_1 - r_2 = \text{speed of the plane against the wind} \]

\[ r_1 + r_2 = \text{speed of the plane with the wind} \]

as shown in Figure 7.10.
Using the formula distance = (rate)(time) for these two speeds, you obtain the following equations.

\[ 2000 = (r_1 - r_2) \left( 4 + \frac{24}{60} \right) \]
\[ 2000 = (r_1 + r_2)(4) \]

These two equations simplify as follows.

\[
\begin{cases} 
5000 = 11r_1 - 11r_2 & \text{Equation 1} \\
500 = r_1 + r_2 & \text{Equation 2}
\end{cases}
\]
To solve this system by elimination, multiply Equation 2 by 11.

\[
\begin{align*}
5000 &= 11r_1 - 11r_2 \\
500 &= r_1 + r_2
\end{align*}
\]

\[
\begin{align*}
5000 &= 11r_1 - 11r_2 \\
5500 &= 11r_1 + 11r_2
\end{align*}
\]

\[
10,500 = 22r_1
\]

Write Equation 1.

Multiply Equation 2 by 11.

Add equations.
Example 8 – Solution

So,

\[ r_1 = \frac{10,500}{22} = \frac{5250}{11} \approx 477.27 \text{ miles per hour} \]

Speed of plane

and

\[ r_2 = 500 - \frac{5250}{11} = \frac{250}{11} \approx 22.73 \text{ miles per hour.} \]

Speed of wind