What You Should Learn

• Recognize partial fraction decompositions of rational expressions.

• Find partial fraction decompositions of rational expressions.
Introduction
Partial fraction decomposition
of \( \frac{x + 7}{x^2 - x - 6} \)

\[
\frac{x + 7}{x^2 - x - 6} = \frac{2}{x - 3} + \frac{-1}{x + 2}. 
\]

Partial fraction  Partial fraction

Introduction

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. Divide if improper: If $N(x)/D(x)$ is an improper fraction [degree of $N(x)$ ≥ degree of $D(x)$], divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = \text{(polynomial)} + \frac{N_1(x)}{D(x)}$$

and apply Steps 2, 3, and 4 below to the proper rational expression $N_1(x)/D(x)$. Note that $N_1(x)$ is the remainder from the division of $N(x)$ by $D(x)$.

2. Factor the denominator: Completely factor the denominator into factors of the form

$$(px + q)^m \quad \text{and} \quad (ax^2 + bx + c)^n$$

where $(ax^2 + bx + c)$ is irreducible.

3. Linear factors: For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of $m$ fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

4. Quadratic factors: For each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of $n$ fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$
Partial Fraction Decomposition
Example 1 – Distinct Linear Factors

Write the partial fraction decomposition of $\frac{x + 7}{x^2 - x - 6}$.

Solution:

$x^2 - x - 6 = (x - 3)(x + 2)$

$$\frac{x + 7}{x^2 - x - 6} = \frac{A}{x - 3} + \frac{B}{x + 2}$$

Write form of decomposition.

$x + 7 = A(x + 2) + B(x - 3)$

Basic equation
Example 1 – Solution

For instance, let \( x = -2 \). Then,

\[
-2 + 7 = A(-2 + 2) + B(-2 - 3)
\]

Substitute \(-2\) for \( x \).

\[
5 = A(0) + B(-5)
\]

\[
5 = -5B
\]

\[-1 = B.\]

To solve for \( A \), let \( x = 3 \) and obtain

\[
3 + 7 = A(3 + 2) + B(3 - 3)
\]

Substitute \(3\) for \( x \).

\[
10 = A(5) + B(0)
\]
Example 1 – Solution

\[ 10 = 5A \]

\[ 2 = A. \]

So, the partial fraction decomposition is

\[ \frac{x + 7}{x^2 - x - 6} = \frac{2}{x - 3} + \frac{-1}{x + 2}. \]
Or...

\[ x + 7 = A(x + 2) + B(x - 3) \]

\[ x + 7 = Ax + 2A + Bx - 3B \]

\[ x + 7 = (A + B)x + (2A - 3B) \]

\[
\begin{cases}
A + B = 1 \\
2A - 3B = 7
\end{cases}
\]

\[
\begin{cases}
A = 2 \\
B = -1
\end{cases}
\]
Partial Fraction Decomposition

**Guidelines for Solving the Basic Equation**

*Linear Factors*

1. Substitute the *zeros* of the distinct linear factors into the basic equation.

2. For repeated linear factors, use the coefficients determined in Step 1 to rewrite the basic equation. Then substitute *other* convenient values of $x$ and solve for the remaining coefficients.

*Quadratic Factors*

1. Expand the basic equation.

2. Collect terms according to powers of $x$.

3. Equate the coefficients of like terms to obtain equations involving $A$, $B$, $C$, and so on.

4. Use a system of linear equations to solve for $A$, $B$, $C$, . . .
**Example**

Write the partial fraction decomposition of \( \frac{8x^3 + 13x}{(x^2 + 2)^2} \)

**Solution:**

It is proper fraction.

\[
\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 3)^2}
\]

\[
\Rightarrow 8x^3 + 13x = (Ax + B)(x^2 + 2) + (Cx + D)
\]

\[
= Ax^3 + 2Ax + Bx^2 + 2B + Cx + D
\]

\[
= Ax^3 + Bx^2 + (2A + C)x + (2B + D)
\]

\[
\Rightarrow \begin{cases} 
A = 8 \\
B = 0 \\
2A + C = 13 \\
2B + D = 0
\end{cases} \quad \Rightarrow \begin{cases} 
A = 8 \\
B = 0 \\
C = -3 \\
C = 0
\end{cases}
\]
Therefore,

\[
\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 3)^2} = \frac{8x}{x^2 + 2} + \frac{-3x}{(x^2 + 3)^2}
\]
Example

Write the partial fraction decomposition of \( \frac{3x^2 + 4x + 4}{x^3 + 4x} \).

Solution:

\( x^3 + 4x = x(x^2 + 4) \)

It is proper fraction

\[
\frac{3x^2 + 4x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}
\]

\[
\Rightarrow 3x^2 + 4x + 4 = A(x^2 + 4) + (Bx + C)x
\]

\[
= (A + B)x^2 + Cx + 4A
\]

\[
\Rightarrow \begin{cases} 
A + B = 3 \\
C = 4 \\
4A = 4
\end{cases} \Rightarrow \begin{cases} 
A = 1 \\
B = 2 \\
C = 4
\end{cases}
\]
Therefore,

\[
\frac{3x^2 + 4x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4} = \frac{1}{x} + \frac{2x + 4}{x^2 + 4}
\]
Partial Fraction Decomposition

\[ \frac{N(x)}{D(x)} = \frac{2x^3 + x^2 - 7x + 7}{x^2 + x - 2} \]