What You Should Learn

• Sketch the graphs of inequalities in two variables.

• Solve systems of inequalities.

• Use systems of inequalities in two variables to model and solve real-life problems.
The Graph of an Inequality
The statements $3x - 2y < 6$ and $2x^2 + 3y^2 \geq 6$ are inequalities in two variables.

An ordered pair $(a, b)$ is a **solution of an inequality** in $x$ and $y$ if the inequality is true when $a$ and $b$ are substituted for $x$ and $y$, respectively.

The **graph of an inequality** is the collection of all solutions of the inequality.
The Graph of an Inequality

To sketch the graph of an inequality, begin by sketching the graph of the corresponding equation.

The graph of the equation will normally separate the plane into two or more regions.

In each such region, one of the following must be true.

1. *All* points in the region are solutions of the inequality.

2. *No* point in the region is a solution of the inequality.
So, you can determine whether the points in an entire region satisfy the inequality by simply testing one point in the region.

### Sketching the Graph of an Inequality in Two Variables

1. Replace the inequality sign by an equal sign, and sketch the graph of the resulting equation. (Use a dashed line for < or > and a solid line for ≤ or ≥.)

2. Test one point in each of the regions formed by the graph in Step 1. If the point satisfies the inequality, shade the entire region to denote that every point in the region satisfies the inequality.
Example 1 – *Sketching the Graph of an Inequality*

Sketch the graph of $y \geq x^2 - 1$.

**Solution:**

Begin by graphing the corresponding equation $y = x^2 - 1$, which is a parabola, as shown in Figure 7.19.

By testing a point *above* the parabola $(0, 0)$ and a point *below* the parabola $(0, -2)$, you can see that the points that satisfy the inequality are those lying above (or on) the parabola.

![Figure 7.19](image_url)
The inequality in Example 1 is a nonlinear inequality in two variables.

Most of the following examples involve linear inequalities such as $ax + by < c$ ($a$ and $b$ are not both zero).

The graph of a linear inequality is a half-plane lying on one side of the line $ax + by = c$. 
Systems of Inequalities
Many practical problems in business, science, and engineering involve systems of linear inequalities. A solution of a system of inequalities in $x$ and $y$ is a point $(x, y)$ that satisfies each inequality in the system.

To sketch the graph of a system of inequalities in two variables, first sketch the graph of each individual inequality (on the same coordinate system) and then find the region that is common to every graph in the system.

This region represents the solution set of the system. For systems of linear inequalities, it is helpful to find the vertices of the solution region.
Example 4 – *Solving a System of Inequalities*

Sketch the graph (and label the vertices) of the solution set of the system.

\[
\begin{align*}
    x - y &< 2 \\
    x &> -2 \\
    y &\leq 3
\end{align*}
\]

Inequality 1
Inequality 2
Inequality 3
Example 4 – Solution

The graphs of these inequalities are shown in Figures 7.22, 7.20, and 7.21, respectively.
Example 4 – Solution

The triangular region common to all three graphs can be found by superimposing the graphs on the same coordinate system, as shown in Figure 7.23.

Figure 7.23
Example 4 – Solution

To find the vertices of the region, solve the three systems of corresponding equations obtained by taking pairs of equations representing the boundaries of the individual regions.

**Vertex A:** \((-2, -4)\)  
\[
\begin{align*}
    x - y &= 2 \\
    x &= -2
\end{align*}
\]

**Vertex B:** \((5, 3)\)  
\[
\begin{align*}
    x - y &= 2 \\
    y &= 3
\end{align*}
\]

**Vertex C:** \((-2, 3)\)  
\[
\begin{align*}
    x &= -2 \\
    y &= 3
\end{align*}
\]

Note in Figure 7.23 that the vertices of the region are represented by open dots. This means that the vertices are \textit{not} solutions of the system of inequalities.
For the triangular region shown in Figure 7.23, each point of intersection of a pair of boundary lines corresponds to a vertex.
With more complicated regions, two border lines can sometimes intersect at a point that is not a vertex of the region, as shown in Figure 7.24.

To keep track of which points of intersection are actually vertices of the region, you should sketch the region and refer to your sketch as you find each point of intersection.
When solving a system of inequalities, you should be aware that the system might have no solution or it might be represented by an unbounded region in the plane.
Applications
Applications

We have discussed the \textit{equilibrium point} for a system of demand and supply equations.

The next example discusses two related concepts that economists call \textit{consumer surplus} and \textit{producer surplus}.

As shown in Figure 7.28, the consumer surplus is defined as the area of the region that lies \textit{below} the demand curve, \textit{above} the horizontal line passing through the equilibrium point, and to the right of the $p$-axis.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure7.28.png}
\caption{Figure 7.28}
\end{figure}
Applications

Similarly, the producer surplus is defined as the area of the region that lies above the supply curve, below the horizontal line passing through the equilibrium point, and to the right of the p-axis.

The consumer surplus is a measure of the amount that consumers would have been willing to pay above what they actually paid, whereas the producer surplus is a measure of the amount that producers would have been willing to receive below what they actually received.
Example 8 – *Consumer Surplus and Producer Surplus*

The demand and supply equations for a new type of personal digital assistant are given by

\[
\begin{align*}
\rho &= 150 - 0.00001x \\ 
\rho &= 60 + 0.00002x
\end{align*}
\]

Demand equation

Supply equation

where \( \rho \) is the price (in dollars) and \( x \) represents the number of units. Find the consumer surplus and producer surplus for these two equations.
Example 8 – Solution

Begin by finding the equilibrium point (when supply and demand are equal) by solving the equation

$$60 + 0.00002x = 150 - 0.00001x.$$  

We have seen that the solution is $x = 3,000,000$ units, which corresponds to an equilibrium price of $p = $120.

So, the consumer surplus and producer surplus are the areas of the following triangular regions.
Example 8 – Solution

**Consumer Surplus**
\[
\begin{align*}
  p &\leq 150 - 0.00001x \\
  p &\geq 120 \\
  x &\geq 0
\end{align*}
\]

**Producer Surplus**
\[
\begin{align*}
  p &\geq 60 + 0.00002x \\
  p &\leq 120 \\
  x &\geq 0
\end{align*}
\]

In Figure 7.29, you can see that the consumer and producer surpluses are defined as the areas of the shaded triangles.

Figure 7.29
Example 8 – Solution

Consumer surplus

\[ = \frac{1}{2} \text{(base)} \times \text{(height)} \]

\[ = \frac{1}{2} \times (3,000,000) \times (30) = $45,000,000 \]

Producer surplus

\[ = \frac{1}{2} \times (\text{base}) \times (\text{height}) \]

\[ = \frac{1}{2} \times (3,000,000) \times (60) = $90,000,000 \]